On the Langlands reciprocity and functoriality principles

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Abstract: I shall describe my reflections on the Langlands reciprocity and functoriality principles.

Let K is a number field. The local Langlands group L_{K_ν} of K_ν is defined by $L_{K_\nu} = WA_{K_\nu} = W_{K_\nu} \times \mathrm{SL}(2,\mathbb{C})$ if $\nu \in \mathbb{h}_K$, and by $L_{K_\nu} = W_{K_\nu}$ if $\nu \in \mathbb{o}_K$, where W_{K_ν} denotes the local Weil group of K_ν. For each $\nu \in \mathbb{h}_K$, fix a Lubin-Tate splitting φ_{K_ν}. The local non-abelian norm-residue homomorphism

$$\{\bullet, K_\nu\}_{\varphi_\nu} : {}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \xrightarrow{\sim} W_{K_\nu}$$

of K_ν is defined and studied in the papers by E. Serbest and the author, where ${}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})}$ is a certain topological group constructed using Fontaine-Wintenberger theory of fields of norms. Fix $\underline{\varphi} = \{\varphi_{K_\nu}\}_{\nu \in \mathbb{h}_K}$ and define the non-commutative topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$, which depends only on K, by the "restricted free topological product"

$$\mathscr{W}\mathscr{A}_K^{\underline{\varphi}} := \mathop{\ast'}_{\nu \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_\nu}^{(\varphi_{K_\nu})} \times \mathrm{SL}(2,\mathbb{C}) : {}_{1}\nabla_{K_\nu}^{(\varphi_{K_\nu})^{\underline{0}}} \times \mathrm{SL}(2,\mathbb{C}) \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}.$$

Here, r_1 and r_2 denote the numbers of real and the pairs of complex conjugate embeddings of the global field K in $\mathbb{C}$. Note that, $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}\,ab} = \mathbb{J}_K$. Let L_K denote the hypothetical Langlands group L_K of K. For $\nu \in \mathbb{h}_K \cup \mathbb{o}_K$, an embedding $e_\nu : K^{sep} \hookrightarrow K_\nu^{sep}$ determines a continuous homomorphism $e_\nu^{\mathrm{Langlands}} : L_{K_\nu} \to L_K$ unique up to conjugacy, which in return defines a continuous homomorphism

$$\mathsf{NR}_{K_\nu}^{(\varphi_{K_\nu})^{\mathrm{Langlands}}} : {}_{\mathbb{Z}}\nabla_{K_\nu}^{\varphi_{K_\nu}} \times \mathrm{SL}(2,\mathbb{C}) \xrightarrow[\sim]{\{\bullet, K_\nu\}_{\varphi_{K_\nu}} \times \mathrm{id}_{\mathrm{SL}(2,\mathbb{C})}} L_{K_\nu} \xrightarrow{e_\nu^{\mathrm{Langlands}}} L_K$$

unique up to conjugacy, for each $\nu \in \mathbb{h}_K$. Fixing one such morphism for each $\nu \in \mathbb{h}_K$, the collection $\{\mathsf{NR}_{K_\nu}^{(\varphi_{K_\nu})^{\text{Langlands}}}\}_{\nu \in \mathbb{h}_K}$ defines a unique continuous homomorphism

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K,$$

which is compatible with Arthur's proposed construction of L_K.

Let G be a connected, quasisplit reductive group over K. There is a bijection between the set of "WA-parameters"

$$\phi : \mathscr{W}\mathscr{A}_K^{\varphi} \to {}^L\mathrm{G}(\mathbb{C}) = \widehat{\mathrm{G}}(\mathbb{C}) \rtimes L_K$$

of G over K and the set $\mathscr{P}_{\mathrm{G}}$ whose elements are the collections

$$\{\phi_\nu : L_{K_\nu} \to {}^L\mathrm{G}_\nu(\mathbb{C})\}_{\nu \in \mathbb{h}_K \cup \mathbb{a}_K}$$

consisting of local L-parameters of G_ν over K_ν for each ν. Note that, assuming the local reciprocity principle for G_ν over K_ν for all $\nu \in \mathbb{h}_K \cup \mathbb{a}_K$, the set $\mathscr{P}_{\mathrm{G}}$ is in bijection with the set whose elements are the collections $\{\Pi_{\phi_\nu}\}_{\nu \in \mathbb{h}_K \cup \mathbb{a}_K}$ of local L-packets of G_ν over K_ν for each ν. As global admissible L-packets of G over K are the restricted tensor products of local L-packets of G_ν over K_ν, by Flath's decomposition theorem, we end up having the following theorems

Theorem 1. *Let* G *be a connected quasisplit reductive group over the number field K. Assume that the local Langlands reciprocity principle for* G *over K holds. Then, there exists a bijection*

$$\{WA\text{-}parameters\ of\ \mathrm{G}\ over\ K\} \leftrightarrow \{global\ admissible\ L\text{-}packets\ of\ \mathrm{G}\ over\ K\}$$

satisfying the "naturality" properties.

and

Theorem 2. *Let* G *and H be connected quasisplit reductive groups over the number field K. Let*

$$\rho : {}^LG \to {}^LH$$

be an L-homomorphism. Assume that the local Langlands reciprocity principle for G *over K holds. Then, the L-homomorphism $\rho : {}^LG \to {}^LH$ induces a map (lifting) from the global admissible L-packets of G over K to the global admissible L-packets of H over K satisfying the "naturality" properties.*

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Time: 13:00

Place: Please contact me for the Google Meet link.